FIRST PASSAGE TIMES AND SOJOURN TIMES FOR BROWNIAN MOTION IN SPACE AND THE EXACT HAUSDORFF MEASURE OF THE SAMPLE PATH

BY

Z. CIESIELSKI AND S. J. TAYLOR(1)

1. **Introduction.** The motivation for carrying out the computations of this paper is to prove a conjecture of Lévy [12]. If $\phi(t)=t^2 \log \log t^{-1}$, then the method of Hausdorff [8] gives a measure $\phi-m(E)$ defined at least for Borel sets E in k-space ($k \geqq 2$). If C_ω denotes the sample path for $0 \leqq t \leqq 1$ of a Brownian motion process in k-space, then Lévy showed that

$$(1.1) \qquad \phi - m(C_\omega) < c,$$

with probability 1, and conjectured that, for $k \geqq 3$,

$$(1.2) \qquad \phi - m(C_\omega) > 0,$$

also with probability 1. This result is proved in §5.

The difficulty in proving lower bounds like (1.2) is that one has to consider all possible coverings of the path by small convex sets in the definition of ϕ-measure. In the past the only successful method has been to use the connection between Hausdorff measures and generalized capacities. The first result of this kind appears in [16] where it is proved that for $k \geqq 2$, the Hausdorff measure with respect to t^α is infinite for all $\alpha < 2$ with probability 1. This result is further sharpened in [7], but there are technical reasons why it is hopeless to obtain exact results like (1.1), (1.2) by capacity arguments. (These technical difficulties are discussed in [17], where it is shown that there is an "uncertainty factor" at least of order $\log t^{-1}$.)

The new method of proof involves the use of a density theorem recently obtained by Taylor and Rogers [14] for general completely additive set functions. In order to apply the density theorem we require the local asymptotic behavior of the path. For $k \geqq 3$ we obtain a "law of the iterated logarithm" for the total time $T_k(a, \omega)$ spent by the path ω in a sphere of radius a as $a \to 0+$. We also determine the local behavior of the first passage time $P_k(a, \omega)$ out of a sphere of radius a for $k \geqq 1$.

In order to obtain the required asymptotic results we required good estimates of the distribution functions for the random variables $T_k(a, \omega)$, $P_k(a, \omega)$. We use the method developed by Mark Kac to compute these dis-

Received by the editors July 9, 1961.

(1) While carrying out this research, Taylor was supported by the National Science Foundation under project NSFG18837.

tributions exactly in §2, 3. It turns out to be possible to write down an exact formula for both distribution functions in terms of an alternating series of negative exponentials (only the first term is important for the asymptotic laws we require in §4). We discovered to our amazement that $T_k(a, \omega)$ and $P_{k-2}(a, \omega)$, $k=3, 4, \cdots$, have precisely the same distribution function. We have not discovered any probabilistic connection between $T_k(a, \omega)$ and $P_{k-2}(a, \omega)$ which explains this phenomenon—it would be interesting if such could be found.

In §6, we discuss briefly the corresponding problems for $k=1, 2$ where we have conjectures but not proofs.

We state now, for convenience of reference, two theorems which we will need in the sequel.

THEOREM A. *Let E_n, $n=1, 2, \cdots$ be a sequence of events such that* $\Pr(E_n) \to 0$ *as* $n \to \infty$, *while* $\sum_{n=1}^{\infty} \Pr(E_n)$ *diverges and* $\Pr(\lim\sup E_n)=0$ *or* 1. *If there exists an absolute constant c such that, for* $n \neq r$,

$$\Pr(E_n \cap E_r) \leqq c \Pr(E_n) \Pr(E_r)$$

then infinitely many of the events E_n occur with probability 1.

This theorem is an immediate consequence of the zero-one law. A proof can be constructed by using a simplified version of the argument on p. 425 of [5].

THEOREM B. *Suppose F is any finite completely additive measure defined for all Borel subsets of Euclidean k-space, h(t) is a continuous monotone increasing function of t with* $\lim_{t\to 0+} h(t)=0$ *and, for any* $\delta>0$,

$$E_\delta = \left\{x: \limsup_{a\to 0} \frac{F(S_k(x,a))}{h(2a)} > \delta\right\},$$

where $S_k(\mathbf{x}, a)$ *denotes the closed sphere center* $\mathbf{x}$, *radius a. Then* E_δ *is a Borel set, and if E is any Borel set with* $E \cap E_\delta = \varnothing$, *then*

$$h - m(E) \geqq \eta_k \delta^{-1} F(E)$$

where $\eta_1, \eta_2, \cdots$ *are absolute constants, and* $h-m(E)$ *denotes the Hausdorff measure of E.*

This is just Lemma 3 of [14] expressed in a notation convenient for the present paper.

2. **Total time spent in a sphere.** We consider the Brownian motion process in k-space starting from the origin, normalized (as usual) in the following way.

Ω denotes the space of paths ω: each $\omega \in \Omega$ corresponds to $\mathbf{x}(t, \omega) = \{x_1(t, \omega), x_2(t, \omega), \cdots, x_k(t, \omega)\}$, $0 \leqq t < \infty$, where $\mathbf{x}(0, \omega)=\mathbf{o}$, and $\mathbf{x}(t, \omega)$

is continuous as a function of t. A measure μ is defined on a field([2]) of measurable subsets of Ω such that if

$$0 < t_1 < \cdots < t_n,$$

and $E_1, E_2, \cdots, E_n$ are any Borel sets of Euclidean k-space the set $Q \subset \Omega$ such that

$$\mathbf{x}(t_1, \omega) \in E_1, \cdots, \mathbf{x}(t_n, \omega) \in E_n$$

is measurable with

$$\mu(Q) = \int_{E_1} \cdots \int_{E_n} \Phi_k(\mathbf{o} \mid \mathbf{r}_1; t_1)\Phi_k(\mathbf{r}_1 \mid \mathbf{r}_2; t_2 - t_1) \cdots \Phi_k(\mathbf{r}_{n-1} \mid \mathbf{r}_n; t_n - t_{n-1}) d\mathbf{r}_1 \cdots d\mathbf{r}_n, \tag{2.1}$$

where each integral is k-dimensional and

$$\Phi_k(\mathbf{r} \mid \boldsymbol{\varrho}; t) = (2\pi t)^{-k/2} \exp\left(-\frac{1}{2} t^{-1} \mid \boldsymbol{\varrho} - \mathbf{r} \mid^2\right), \tag{2.2}$$

where $|\mathbf{r}|$ denotes the Euclidean distance of $\mathbf{r}$ from the origin $\mathbf{o}$. Then $\mu(\Omega) = 1$, so we have a probability measure. Let $S_k = S_k(a)$ denote the solid (closed) sphere in k-space with center at $\mathbf{o}$ and radius a. The characteristic function of S_k will be denoted by

$$V_k(\mathbf{r}; a) = \begin{cases} 1 & \text{if } |\mathbf{r}| \leqq a; \\ 0 & \text{if } |\mathbf{r}| > a. \end{cases}$$

In the present section we will consider only the case of $k \geqq 3$. It is well known that, with probability 1, we have $|\mathbf{x}(t, \omega)| \to \infty$ as $t \to \infty$ for $k \geqq 3$. (Exact estimates of the rate of escape to infinity of the Brownian path are given by Dvoretsky-Erdös [4].) Let $T_k(a, \omega)$ be the total time spent in the sphere S_k by the path. Then T_k is a random variable satisfying

$$T_k(a; \omega) = \int_0^\infty V_k(\mathbf{r}(\tau); a) d\tau \tag{2.3}$$

and T_k is finite with probability 1.

Let $F_k(x, a)$ denote the distribution function of the random variable $T_k(a, \omega)$. Thus

$$F_k(x, a) = \mu\{T_k(a, \omega) < x\}. \tag{2.4}$$

In order to compute $F_k(x, a)$ exactly we use methods developed and used extensively by M. Kac (see [10], for example). These consist essentially in computing the Laplace transform of the distribution function $F_k(x, a)$ and then inverting. In order to obtain integral equations and differential equations

([2]) Measurability problems will not be of importance in this paper. The method for obtaining the measure μ is described in Doob [3].

whose solution will be the required transform we need to consider a slightly more general situation.

For $y \in R_k$, let

$$T_k(a, y; \omega) = \int_0^\infty V_k(x(\tau, \omega) + y; a) d\tau \tag{2.5}$$

and put

$$h_k(y; u) = E\{\exp[-uT_k(a, y; \omega)]\} = \int_0^\infty e^{-ux} dF_k(x, a; y) \tag{2.6}$$

where $F_k(x, a; y)$ is the distribution function of the total time spent in S_k by a Brownian motion process starting from y; and $E\{ \ \}$ denotes the expectation of the random variable between the braces.

To compute $h_k(y; u)$ we use the moments of the distribution. Using (2.1), (2.2) and a standard transformation we have

$$\begin{aligned} \mu_n(y) &= E\{T_k^n(a, y; \omega)\} \\ &= n! \int_{0<\tau_1<\cdots<\tau_n<\infty} d\tau_1 \cdots \int d\tau_n \int_{S_k} \cdots \int_{S_k} \Phi_k(y \mid r_1; \tau_1) \cdots \\ &\qquad \Phi_k(r_{n-1} \mid r_n; \tau_n - \tau_{n-1}) dr_1 \cdots dr_n. \end{aligned} \tag{2.7}$$

Since for $t > 0$ we have

$$\int_t^\infty \Phi_k(y \mid x; \tau - t) d\tau = \frac{2^\nu \Gamma(\nu)}{(2\pi)^{\nu+1}} \frac{1}{|x - y|^{k-2}},$$

where $\nu = (k-2)/2$, (2.7) can be written as

$$\begin{aligned} \mu_n(y) &= n! \left(\frac{2^\nu \Gamma(\nu)}{(2\pi)^{\nu+1}} \right)^n \int_{S_k} \cdots \int_{S_k} \frac{1}{|r_1 - y|^{k-2}} \cdots \\ &\qquad \frac{1}{|r_n - r_{n-1}|^{k-2}} dr_1, dr_2, \cdots dr_n. \end{aligned} \tag{2.8}$$

In connection with (2.8) we consider the integral equation

$$\int_{S_k} \phi(\varrho) K(\varrho, r) d\varrho = \lambda \phi(r), \qquad r \in S_k \tag{2.9}$$

where

$$K(\varrho, r) = \frac{2^\nu \Gamma(\nu)}{(2\pi)^{\nu+1}} \frac{1}{|\varrho - r|^{k-2}}. \tag{2.10}$$

The kernel $K(\varrho, r)$ has finite Hilbert-Schmidt norm for $k = 3$, but not for $k > 3$. However the s-fold iterated kernel $K^{(s)}(\varrho, r)$ is bounded for $s \geqq k$, and so certainly has finite Hilbert-Schmidt norm. Further $K(\varrho, r)$ is symmetric and it is not difficult to show that it is the kernel of an integral operator which is both positive definite and completely continuous. It follows from the theory of integral equations(3) that

(i) there are countably many positive eigenvalues

$$\lambda_1 \geqq \lambda_2 \geqq \cdots > 0;$$

and corresponding eigenfunctions (which we assume to be normalised)

$$\phi_1, \phi_2, \cdots;$$

of the integral equation (2.9):

(ii) the integral equation with kernel $K^{(s)}(\varrho, r)$ has the same eigenfunctions ϕ_i with corresponding eigenvalues λ_i^s, for all integer values of s:

$$\text{(iii)} \qquad K^{(s)}(\varrho, r) = \sum_{i=1}^{\infty} \lambda_i^s \phi_i(\varrho)\phi_i(r), \qquad s \geqq k$$

and this series converges uniformly and absolutely in ϱ and r:

(iv) the eigenfunctions ϕ_i form a complete system of functions in $L^2(S_k)$:

(v) $\phi_1(r) > 0$ for all $r \in S_k$, and $\lambda_1 > \lambda_2$; since $\phi_2(r)$ has to take both positive and negative values on sets of positive measure, and $K(\varrho, r) > 0$ for all $\varrho, r \in S_k$.

From (2.8) we deduce that

$$\frac{\mu_n(y)}{n!} = \int_{S_k} K^{(n)}(r, y)dr$$

from which it follows that

$$\text{(2.11)} \qquad \frac{\mu_n(y)}{n!} = \sum_{j=1}^{\infty} \lambda_j^n c_j \phi_j(y) \qquad \text{for } n \geqq k, y \in S_k,$$

where $c_j = \int_{S_k} \phi_j(\varrho)d\varrho$; and

$$\text{(2.12)} \qquad \int_{S_k} K(r, y) \frac{\mu_n(y)}{n!} dy = \frac{\mu_{n+1}(r)}{(n+1)!}.$$

Using remarks (iii) and (v) above, we deduce from (2.11) that

$$\frac{\mu_n(y)}{n!} \sim c_1\phi_1(y)\lambda_1^n, \qquad \text{as } n \to \infty$$

uniformly in y for $y \in S_k$. It follows that

$$h_k(y; u) = \sum_{n=0}^{\infty} (-1)^n \frac{\mu_n(y)}{n!} u^n$$

(3) See, for example [1, Vol. 1].

converges uniformly in $y \in S_k$ for each fixed u with $|u| < 1/\lambda_1$. We may apply the integral operator term by term to obtain the integral equation

$$\int_{S_k} K(y, r) h_k(y; u) dy = \frac{1}{u} (1 - h_k(r; u)) \tag{2.13}$$

valid for $r \in S_k$ and $|u| < \lambda_1^{-1}$.

Now if we apply the k-dimensional Laplace operator to both sides of (2.13) and use Green's theorem in k-space we obtain the differential equation

$$\Delta h_k(r; u) = 2u h_k(r; u) \tag{2.14}$$

for $r \in S_k$ and $|u| < \lambda_1^{-1}$.

But for each fixed r, $h_k(r; u)$ is an analytic function of u at least for $R(u) > 0$. We have seen it is also analytic for u real with $-\lambda_1^{-1} < u < \lambda_1^{-1}$. It follows that it must be analytic for all u with $R(u) > \lambda_1^{-1}$, and (2.14) must be valid at least in this range.

In the present note we are interested only in $h_k(o; u)$ so that we do not need to compute $h_k(r; u)$ for r outside S_k. However it is worth remarking that, outside S_k, $h_k(r; u)$ is a harmonic function of r. Further it can be easily shown that $h_k(r; u) \to 1$ as $|r| \to \infty$. These facts allow one to compute the function outside S_k as soon as it is known inside.

It is clear that $h_k(r; u)$ has spherical symmetry. The fundamental solution of (2.14) which is bounded in S_k, and depends only on $|r|$ is of the form (see, for example [1, Vol. II, p. 227])

$$h_k(r, u) = \frac{\gamma(u)}{|r|^\nu} J_\nu(i(2u)^{1/2} |r|), \tag{2.15}$$

where $J_\nu(z)$ denotes the Bessel function of the first kind of order ν, and $\nu = (k-2)/2$.

To determine $\gamma(u)$ we substitute (2.15) into the integral equation (2.13). This gives

$$\frac{1}{\gamma(u)} = u \int_{S_k} \frac{J_\nu(i(2u)^{1/2}|y|)}{|y|^\nu} K(y, r) dy + \frac{J_\nu(i(2u)^{1/2}|r|)}{|r|^\nu} \qquad \text{for } r \in S_k.$$

The right-hand side must be independent of r, so we may substitute $r = o$ and use the series expansion of $J_\nu(z)$. This gives

$$\begin{aligned} \frac{1}{\gamma(u)} &= u \frac{\Gamma(\nu)}{(2\pi)^{k/2}} \int_{S_k} \frac{J_\nu(i(2u)^{1/2}|y|)}{|y|^{(3/2)(k-2)}} dy + \frac{(i(2u)^{1/2})^\nu}{2^\nu \Gamma(k/2)} \\ &= u \frac{\Gamma(\nu)}{\Gamma(\nu + 1)} \int_0^a r^{2-k/2} J_\nu(i(2u)^{1/2}|r|) dr + \frac{(i(2u)^{1/2})^\nu}{2^\nu \Gamma(k/2)} . \end{aligned} \tag{2.16}$$

Substituting $r = o$ in (2.15) gives

$$h_k(o,u) = \gamma(u)\frac{(i(2u)^{1/2})^\nu}{2^\nu\Gamma(k/2)}$$

so that, by (2.16) we have

$$\begin{aligned}\frac{1}{h_k(o,u)} &= 1 + u\frac{2^\nu\Gamma(\nu)}{(i(2u)^{1/2})^\nu}\int_0^a r^{2-k/2}J_\nu(i(2u)^{1/2}r)dr \\ &= 1 - \frac{z^2}{2}\frac{\Gamma(\nu)}{(z/2)^\nu}\int_0^{\pi/2} J_\nu(z\sin\theta)(\sin\theta)^{1-\nu}\cos\theta d\theta\end{aligned} \tag{2.17}$$

where $z=ia(2u)^{1/2}$. Using the identity (2) from p. 374 of [18], where $s_{m,n}$ is defined on p. 345 of [18], one gets

$$\int_0^{\pi/2} J_\nu(z\sin\theta)(\sin\theta)^{1-\nu}\cos\theta d\theta = \frac{z^\nu}{2^{\nu-1}z^2}\sum_{m=0}^\infty\frac{(-1)^m(z/2)^{2m+2}}{(m+1)!\Gamma(\nu+m+1)}. \tag{2.18}$$

Now substituting (2.18) into (2.17) gives

$$\begin{aligned}\frac{1}{h_k(o,u)} &= 1 - \Gamma(\nu)\sum_{m=0}^\infty\frac{(-1)^m(z/2)^{2m+2}}{(m+1)!\Gamma(\nu+m+1)} \\ &= \Gamma(\nu)\sum_{m=0}^\infty\frac{(-1)^m(z/2)^{2m}}{m!\Gamma(\nu+m)}.\end{aligned}$$

Thus we obtain the formula

$$\frac{1}{h_k(o,u)} = \frac{\Gamma(\nu)}{(z/2)^{\nu-1}}J_{\nu-1}(z). \tag{2.19}$$

Using the representation of Bessel functions as an infinite product [18, p. 498], we have, on substituting $z=ia(2u)^{1/2}$,

$$h_k(o,u) = \frac{1}{\prod_{r=1}^\infty\left(1+\frac{2a^2u}{j_{k,r}^2}\right)} \tag{2.20}$$

where $\{j_{k,r}\}$ $r=1, 2, \cdots$ are the positive zeros in order of magnitude of the Bessel function $J_{\nu-1}(z)$.

It is worth remarking at this point that the formula (2.19) is particularly simple when $k=3$. It reduces to

$$h_3(o,u) = \frac{1}{\cos(ia(2u)^{1/2})}.$$

Using the Mittag-Leffler expansion of $1/\cos z$ considered as a meromorphic function, this gives

$$h_3(o, u) = \frac{\pi}{2a^2} \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{u + (2n+1)^2\pi^2/8a^2}. \tag{2.21}$$

We may invert this Laplace Transform term by term to give

$$1 - F_3(x, a) = \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \exp\left(-\frac{(2n+1)^2\pi^2}{8a^2} x\right). \tag{2.22}$$

We were surprised to notice that $F_3(x, a)$ turns out to be the distribution for the maximum displacement of a one-dimensional Brownian motion in a fixed time interval. The series (2.22) converges very rapidly for large x, but not for very small positive x. It is worth remarking that another method of inverting $h_3(o, u)$ leads to the density function $p_3(x)$ for the distribution $F_3(x, a)$ given by

$$p_3(x) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)a}{x^{3/2}} \exp\left(-\frac{a^2(2n+1)^2}{2x}\right),$$

and this series converges fast for small positive x, but very slowly for large x.

For $k>3$, though the series corresponding to (2.21) can be written down (and even inverted to give the correct answer!) it does not converge as the terms do not even tend to zero. We have to use different machinery to invert (2.20) in general.

Consider now the following density function

$$p_k(x) = \frac{1}{2^\nu \Gamma(\nu) a^2} \sum_{r=1}^{\infty} \frac{j_{k,r}^{\nu}}{J_\nu(j_{k,r})} \exp\left(-\frac{j_{k,r}^2}{2a^2} x\right), \qquad x > 0 \tag{2.23}$$

where $\nu=(k-2)/2$. Our object now is to show that $h_k(o, u)$ is the Laplace transform of $p_k(x)$.

Put

$$H_k^{(n)}(u) = \prod_{r=1}^{n} \left(1 + \frac{2a^2u}{j_{k,r}^2}\right)^{-1}.$$

Then (modify [9, Theorem 8.2, p. 60]), if $p_k^{(n)}(x)$ has $H_k^{(n)}(u)$ for its Laplace transform, and we can show that

$$\lim_{n\to\infty} p_k^{(n)}(x) = p_k(x), \qquad \text{for } x > 0, \tag{2.24}$$

then $p_k(x)$ is the result of inverting $h(o, u)$.

To determine $p_k^{(n)}(x)$, express $H_k^{(n)}(u)$ in partial fractions. Thus

$$H_k^{(n)}(u) = \sum_{r=1}^{n} \frac{\alpha_{k,n,r}}{u + j_{k,r}^2/2a^2}$$

where

$$\alpha_{k,n,r} = \frac{j_{k,r}^2}{2a^2} \prod_{s\neq r;\, s=1}^{n} \left(1 - \frac{j_{k,r}^2}{j_{k,s}^2}\right)^{-1}. \tag{2.25}$$

Hence we have

$$p_k^{(n)}(x) = \sum_{r=1}^{n} \alpha_{k,n,r} \exp\left(-\frac{j_{k,r}^2}{2a^2} x\right). \tag{2.26}$$

Since the infinite product converges we may write

$$\begin{aligned}\beta_{k,r} = \lim_{n\to\infty} \alpha_{k,n,r} &= \frac{j_{k,r}^2}{2a^2} \prod_{s\neq r;\, s=1}^{\infty} \left(1 - \frac{j_{k,r}^2}{j_{k,s}^2}\right)^{-1} \\ &= -\frac{j_{k,r}^2}{2a^2} \frac{(j_{k,r}/2)^{\nu-2}}{\Gamma(\nu)} \cdot \frac{1}{J_{\nu'-1}(j_{k,r})}\end{aligned}$$

using the infinite product (3) on page 498 of [18] and noting that it converges uniformly for z in a compact set. Since $j_{k,r}$ is a zero of $J_\nu(z)$ we can deduce from the standard recurrence relation that

$$\beta_{k,r} = \frac{j_{k,r}^{\nu}}{2^\nu \Gamma(\nu) a^2} \frac{1}{J_\nu(j_{k,r})}.$$

Thus $\beta_{k,r}$ are the coefficients in the series (2.23) for $p_k(x)$. It is worth noting that the coefficients $\{\beta_{k,r}\}$ $r=1, 2, \cdots$ alternate in sign because of the term $J_\nu(j_{k,r})$. One can show easily that

$$\beta_{k,r} = O(r^{\nu+1/2}),$$

so that there is no difficulty about the convergence of (2.23). A simple argument will now suffice to prove (2.24) so that we have obtained the following.

THEOREM 1. *Let $T_k(a;\omega)$ be the total time spent in a sphere center* o, *radius* a *by a Brownian path in k-space* $k \geqq 3$ *which starts from the origin, then*

$$\mu\{T_k(a,\omega) > x\} = \sum_{r=1}^{\infty} \psi_{k,r} \exp\left(-\frac{p_{k,r}^2}{2a^2} x\right)$$

where $\{p_{k,r}\}$ $r=1, 2, \cdots$ *are the positive zeros of the Bessel function* $J_\mu(z)$ *with* $\mu = k/2-2$, *and*

$$\psi_{k,r} = \frac{1}{2^{\mu-1}\Gamma(\mu+1)} \frac{p_{k,r}^{\mu-1}}{J_{\mu+1}(p_{k,r})}.$$

Proof. All that remains to be done is to integrate $p_k(t)$ given by (2.23) from x to ∞.

REMARK. For $k=5$ the distribution again has a very simple form.

3. **First passage time out of a sphere.** We again consider Brownian motion in k-space starting from the origin, but this time k can be any positive integer. Let $P_k(a, \omega)$ be the infinum of the values of t for which $\mathbf{x}(t, \omega)$ is outside $S_k = S_k(a)$. Then $P_k(a, \omega)$ is a random variable and

$$\{P_k(a, \omega) > x\} = \left\{\sup_{0 \leq t \leq x} |\mathbf{x}(t, \omega)| < a\right\}.$$

Put

$$G_k(x, a) = \mu\{P_k(a, \omega) > x\} = \mu\left\{\sup_{0 \leq t \leq x} |\mathbf{x}(t, \omega)| < a\right\}. \tag{3.1}$$

The form of the function $G_k(x, a)$ was obtained by Lévy in [12], where he obtains the following expression for (3.1).

$$G_k(x, a) = \sum_{r=1}^{\infty} \xi_{k,r} \exp\left(-\frac{q_{k,r}^2}{2a^2} x\right) \tag{3.2}$$

where $q_{k,r}$ are the positive zeros of the Bessel function $J_\nu(z)$ with $\nu = (k-2)/2$, and the coefficients $\xi_{k,r}$ are not determined explicitly. Using the results of Lévy we see that

$$\xi_{k,r} = \frac{\int_{S_k} \phi_{k,r}(y)dy}{\int_{S_k} \phi_{k,r}^2(y)dy} \phi_{k,r}(o) \tag{3.3}$$

where

$$\phi_{k,r}(y) = 2^\nu\Gamma(\nu+1)\left(\frac{a}{q_{k,r}|y|}\right)^\nu J_\nu\left(\frac{|y| q_{k,r}}{a}\right).$$

Substituting in (3.3) and transforming to polar coordinates gives

$$\xi_{k,r} = \frac{1}{2^\nu\Gamma(\nu+1)} \frac{\int_0^{q_{k,r}} t^{\nu+1}J_\nu(t)dt}{\int_0^{q_{k,r}} tJ_\nu^2(t)dt}. \tag{3.4}$$

Using the standard recurrence relation (5) of [18, p. 46] to evaluate the numerator, and relation (11) of [18, p. 135] to evaluate the denominator; we obtain

$$\xi_{k,r} = -\frac{1}{2^\nu\Gamma(\nu+1)} \frac{q_{k,r}^{\nu+1} J_{\nu+1}(q_{k,r})}{\frac{1}{2} q_{k,r}^2 J_{\nu-1}(q_{k,r}) J_{\nu+1}(q_{k,r})}$$

$$= \frac{q_{k,r}^{\nu-1}}{2^{\nu-1}\Gamma(\nu+1)} \frac{1}{J_{\nu+1}(q_{k,r})};$$

using the fact that $J_{\nu-1}(q_{k,r}) = -J_{\nu+1}(q_{k,r})$ since $q_{k,r}$ is a zero of $J_\nu(t)$.

We have therefore proved by computation the following

THEOREM 2. *If $T_k(a, \omega)$ denotes the total time spent in a sphere of radius a center* **o** *by a Brownian path in k-space, and $P_k(a, \omega)$ denotes the first passage time out of the sphere for the same path then $T_{k+2}(a, \omega)$ and $P_k(a, \omega)$ have precisely the same distribution for $k=1, 2, \cdots$. This distribution is given by*

$$\mu\{P_k(a, \omega) > x\} = \sum_{r=1}^{\infty} \xi_{k,r} \exp\left(-\frac{q_{k,r}^2}{2a^2} x\right)$$

where $q_{k,r}$ are the positive roots of the Bessel function $J_\nu(z)$ with $\nu = k/2-1$ and

$$\xi_{k,r} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{q_{k,r}^{\nu-1}}{J_{\nu+1}(q_{k,r})}.$$

REMARK 1. It is possible to compute the distribution function for $P_k(a, \omega)$ by using similar methods to those we used in Theorem 1; (see Rosenblatt [15], where related computations are carried out). We refrained from writing down the details of this method when we discovered that Lévy had already computed it by slightly different methods except for the evaluation of the constants.

REMARK 2. An immediate question is raised by Theorem 2: viz. is there any probabilistic reason why T_{k+2} and P_k should have the same distribution? It would be nice to obtain a proof for Theorem 2 which did not involve computing both distributions. We have no idea how such a proof can be constructed.

REMARK 3. Theorem 2 is not true for regions more general than the sphere; nor is it true for the sphere if the Brownian path starts at any point other than the center.

REMARK 4. It is interesting to compare our distribution function with the Kolmogorov-Smirnov distribution in k-space which was computed by Kiefer in [11]. The coefficients $(-q_{k,r}^2/2a^2)$ are the same while the $\xi_{k,r}$ are different but related.

4. **Local asymptotic laws.** We now have a number of random variables:

(i) $T_k(a, \omega)$ whose expectation is $c_{k,1}a^2$ $(k \geqq 3)$;

(ii) $P_k(a, \omega)$ whose expectation is $c_{k,2}a^2$ $(k \geqq 1)$;

(iii) $R_k(t, \omega)$ $\sup_{0 \leq \tau \leq t} |x(\tau, \omega)|$ whose expectation is $c_{k,3}t^{1/2}$ $(k \geqq 1)$ for suitable constants, $c_{k,1}$, $c_{k,2}$, $c_{k,3}$.

The random variables (ii) and (iii) are clearly related by

$$P_k(a, \omega) > t \tag{4.1}$$

if and only if $R_k(t, \omega) < a$. The classical law of the iterated logarithm refers to the large values of $R_k(t, \omega)$ as t increases to ∞. For the purposes of application we are interested in the large values of $a^{-2}T_k(a, \omega)$ as $a \to 0+$. Our object is to prove

THEOREM 3. *For almost all* $\omega \in \Omega$, $k \geqq 3$

$$\limsup_{a \to 0+} \frac{T_k(a, \omega)}{a^2 \log \log a^{-1}} = \frac{2}{p_k^2}$$

where p_k *is the first positive zero of* $J_\mu(z)$ *with* $\mu = k/2 - 2$.

Very similar arguments suffice to prove

THEOREM 4. *For almost all* $\omega \in \Omega$, $k \geqq 1$

$$\text{(i)} \qquad \limsup_{a \to 0+} \frac{P_k(a, \omega)}{a^2 \log \log a^{-1}} = \frac{2}{q_k^2},$$

$$\text{(ii)} \qquad \liminf_{t \to 0+} \frac{R_k^2(t, \omega)}{t \log \log t^{-1}} = \frac{q_k^2}{2},$$

where q_k *is the first positive zero of* $J_\nu(z)$ *with* $\nu = k/2 - 1$.

It is possible to deduce Theorem 4(i) from Theorem 4(ii) by using (4.1), and it is also possible to deduce Theorem 4(ii) from the results of pages 30–33 of Lévy [12]. Therefore we will not give any proof of Theorem 4 in the present paper.

Proof of Theorem 3. From Theorem 1 it follows that

$$\mu\{T_k(a, \omega) > x\} = d_k \exp\left(-\frac{p_k^2}{2a^2}\, x\right)[1 + O(\exp(-\mu_k x))] \tag{4.2}$$

for suitable positive constants d_k, μ_k where p_k is the first positive root of $J_\mu(z)$, $\mu = k/2 - 2$. Because of independence difficulties we need to consider the random variable

$$T_k(a, X, \omega) = \int_0^X V_k(r(t); a)dt \tag{4.3}$$

which is the total time spent in S_k up to time X. We need to make precise the idea that the distribution of $T_k(a, X, \omega)$ is very close to that of $T_k(a, \omega)$ for large X. Put

$$Q_k(a, X) = \mu\{\mathrm{x}(t, \omega) \in S_k(a) \text{ for some } t \geqq X\}. \tag{4.4}$$

Then it is proved by Dvoretzky and Erdös [5, Lemma 2] that, for $k \geqq 3$,

$$s_k\left(\frac{a}{X}\right)^{k-2} e^{-a^2/2X} \leqq Q_k(a, X) \leqq s_k\left(\frac{a}{X}\right)^{k-2} \tag{4.5}$$

where

$$s_k = \frac{k}{2^{k/2}\Gamma(k/2+1)}.$$

It is clear that

$$\mu\{T_k(a, \omega) > x\} - Q_k(a, X) \leqq \mu\{T_k(a, X, \omega) > x\} \leqq \mu\{T_k(a, \omega) > x\}. \tag{4.6}$$

Consider now the monotone sequence

$$b_n = e^{-n(\log n)}, \qquad n = 1, 2, \cdots. \tag{4.7}$$

Put

$$X_n = b_n^2 n^3. \tag{4.8}$$

Let E_n denote the event

$$T_k(b_n, X_n, \omega) > \left(\frac{2}{p_k^2} - \epsilon\right) b_n^2 \log\log(1/b_n) = t_{k,n}. \tag{4.9}$$

Using (4.6), (4.5), (4.7), (4.8) and (4.2) we obtain

$$\mu(E_n) = d_k\{n \log n\}^{-1+\epsilon p_k^2/2}(1 + o(1))$$

as $n \to \infty$. Thus for each $\epsilon > 0$, the series $\sum_{n=1}^{\infty} \mu(E_n)$ diverges. Although the events E_n are not independent they are almost so. The easiest way to see this is to consider $E_n \cap E_{n+r}$. This event can only happen if E_{n+r} happens and in addition a Brownian path starting from $\mathrm{x}(X_{n+r}, \omega)$ and proceeding for time $(X_n - X_{n+r})$ spends at least $(t_{k,n} - X_{n+r})$ time in $S_k(b_n)$. Thus

$$\begin{aligned} \mu(E_n \cap E_{n+r}) &\leqq \mu(E_{n+r})\mu\{T_k(b_n, \omega) > t_{k,n} - X_{n+1}\} \\ &= \mu(E_{n+r})\,\mu(E_n)(1 + o(1)); \end{aligned}$$

since $n^{\exp[-c(\log n)^2]} \to 1$ as $n \to \infty$, on applying (4.7), (4.8) and (4.2).

Since the zero-one law of probability applies to the sequence E_n, we may now apply Theorem A to deduce that, with probability 1 infinitely many of the events E_n occur. Hence

$$\limsup_{a\to 0+} \frac{T_k(a,\omega)}{a^2 \log\log(1/a)} \geqq \frac{2}{p_k^2}. \tag{4.10}$$

Now consider the sequence

$$a_n = e^{-n/\log n}, \qquad n = 1, 2, \cdots. \tag{4.11}$$

Let

$$F_n = \left\{\omega: T_k(a_n, \omega) > (2p_k^{-2} + 2\epsilon)a_{n+1}^2 \log\log(1/a_{n+1})\right\}. \tag{4.12}$$

By (4.11) for each $\epsilon > 0$, there is an n_0 such that

$$F_n \subset G_n \qquad \text{for } n \geqq n_0, \tag{4.13}$$

where

$$G_n = \left\{\omega: T_k(a_n, \omega) > (2p_k^{-2} + \epsilon)a_n^2 \log\log(1/a_n)\right\}.$$

By (4.2),

$$\mu(G_n) = d_k\{n/\log n\}^{-1-\epsilon p_k^2/2}(1 + o(1)).$$

Hence $\sum_{n=1}^{\infty}\mu(G_n)$ converges and, by (4.13) this implies that $\sum_{n=1}^{\infty}\mu(F_n)$ converges. Applying the Borel-Cantelli lemma in the other direction we see that, for almost all ω, there exists an integer $N(\omega)$ such that F_n does not occur for $n \geqq N(\omega)$. If $a_n \leqq a \leqq a_{n+1}$, $n \geqq N(\omega)$,

$$\frac{T_k(a,\omega)}{a^2 \log\log(1/a)} < \frac{T_k(a_n,\omega)}{a_{n+1}^2 \log\log(1/a_{n+1})} < \frac{2}{p_k^2} + 2\epsilon.$$

Since ϵ is arbitrary it follows that, with probability 1,

$$\limsup_{a\to 0+} \frac{T_k(a,\omega)}{a^2 \log\log(1/a)} \leqq \frac{2}{p_k^2}.$$

This together with (4.10) completes the proof of Theorem 3.

REMARK. $p_3 = \pi/2$, so that the case $k = 3$ of Theorem 3 reads

$$\limsup_{a\to 0+} \frac{T_3(a,\omega)}{a^2 \log\log(1/a)} = \frac{8}{\pi^2} \text{ with probability 1.}$$

5. **The exact Hausdorff measure function for Brownian paths in k-space, $k \geqq 3$.** For each $\omega \in \Omega$, we can think of the set of points $\mathbf{x}(t, \omega)$, $0 \leqq t \leqq 1$ as a point set C_ω in Euclidean k-space. Since $\mathbf{x}(t, \omega)$ is continuous we may define a set

function $F_\omega(E)$ for every Borel set E in Euclidean k-space by

$$F_\omega(E) = m(E(\omega)) \tag{5.1}$$

where $E(\omega)$ is the set of t such that $\mathbf{x}(t, \omega) \in E$, $0 \leqq t \leqq 1$ and $m(\ )$ denotes the Lebesgue linear measure. The set function $F_\omega(E)$ is thus non-negative, completely additive and defined at least for all Borel sets. We may apply the analysis of Rogers and Taylor [13; 14] to $F_\omega(E)$. In particular, let

$$\phi(t) = t^2 \log \log t^{-1} \tag{5.2}$$

and let us calculate

$$\overline{D}_\phi F_\omega(\mathbf{x}) = \limsup_{a \to 0+} \frac{F_\omega(S_k(\mathbf{x}, a))}{\phi(2a)} \cdot \tag{5.3}$$

If $\mathbf{x} \neq \mathbf{x}(t, \omega)$ for any t, $0 \leqq t \leqq 1$, then since the path is a closed set we have

$$\overline{D}_\phi F_\omega(\mathbf{x}) = 0. \tag{5.4}$$

If $\mathbf{x} = \mathbf{x}(t_0, \omega)$ then with probability 1, for each $0 \leqq t_0 \leqq 1$,

$$\limsup_{a \to 0+} \frac{T_k(a, \omega)}{4a^2 \log \log(1/a)} \leqq \overline{D}_\phi F_\omega(\mathbf{x}) \leqq 2 \limsup_{a \to 0} \frac{T_k(a, \omega)}{4a^2 \log \log(1/a)} = \frac{1}{p_k^2}$$

on using Theorem 3 with (5.3), (5.2), and (5.1). (In the above inequality the lower limit for $\overline{D}_\phi F_\omega(\mathbf{x})$ is almost always attained: we do not prove this as we require only an upper bound.)

By setting up a product measure in $[0, 1] \times \Omega$ and applying Fubini's theorem, it follows that, with probability 1,

$$\overline{D}_\phi F_\omega(\mathbf{x}) \leqq p_k^{-2} \tag{5.5}$$

for almost all t_0 in $[0, 1]$ where $\mathbf{x} = \mathbf{x}(t_0, \omega)$. Hence F_ω is concentrated on a subset $D_\omega \subset C_\omega$ such that (5.5) is satisfied for each $\mathbf{x} \in D_\omega$, and (5.4) is satisfied for each $\mathbf{x}$ not in C_ω. It follows from Theorem B that

$$\phi - m(D_\omega) \geqq \eta_k p_k^2 > 0.$$

Thus we have proved that, with probability 1, the ϕ-measure of the path set C_ω is positive. In [12], by applying a suitable version of Theorem 4(ii), Lévy proved that with probability 1, the ϕ-measure of C_ω is also finite. Lévy [12] also pointed out that, under these conditions, an application of the zero-one law now shows the existence of absolute constants ζ_k ($k = 3, 4, \cdots$) such that

$$\phi - m(C_\omega) = \zeta_k.$$

We state as our final result

THEOREM 5. *For $k=3, 4, \cdots$ let $C_\omega(t)$ denote the set of points in k-space which are on the k-dimensional Brownian motion $\mathbf{x}(t, \omega)$, $0 \leqq \tau \leqq t$. Then there are constants ζ_k such that, with probability* 1,

$$\phi - m\{C_\omega(t)\} = \zeta_k t$$

where $\phi - m(\ )$ denotes the Hausdorff measure with respect to the function $\phi(t) = t^2 \log \log t^{-1}$.

6. **Problems for $k=1, 2$.** The random variable $T_k(a, \omega)$ is not defined for $k=1, 2$ because it is infinite with probability 1. However it is still reasonable to study $T_k(a, X, \omega)$ defined by (4.3). The methods of computation have to be changed, and we did not succeed in finding the exact form of the distribution. Kac and Darling [2] obtained the first term of the asymptotic expansion for $T_2(a, X, \omega)$ which we can write

$$\lim_{X\to\infty} \mu\left\{\frac{2}{a^2 \log X} T_2(a, X, \omega) < \alpha\right\} = 1 - e^{-\alpha}. \tag{6.1}$$

By a transformation of scale this can be written

$$\lim_{a\to 0+} \mu\left\{\frac{1}{a^2 \log(1/a)} T_2(a, 1, \omega) < \alpha\right\} = 1 - e^{-\alpha}. \tag{6.2}$$

In order to carry out the computations of §4 for $k=2$, one would require additional terms in the asymptotic expansion. It is likely that our methods would yield such terms but the computational difficulties are considerable.

Even if one had a sufficiently good estimate for $\mu\{T_2(a, 1, \omega) > x\}$, there is a further difficulty in carrying out the details of §4. This is caused by independence troubles in the application of Borel-Cantelli's lemma. Our *conjecture* is that

$$\limsup_{a\to 0+} \frac{T_2(a, 1, \omega)}{a^2 \log(1/a) \log\log\log(1/a)} = 1, \tag{6.3}$$

with probability 1(4). This is suggested by the similarity between (6.2) and the distribution obtained in [6] for the number of returns to the origin of a Pólya random walk in the plane.

If (6.3) were true then one would further conjecture that the h-measure of C_ω for a Brownian path in the plane must be a constant, ϕ_2 where $h(t) = t^2 \log t^{-1} \log\log\log t^{-1}$. We used the methods of random walk approximation in [7] to consider the Hausdorff measure of the plane Brownian path—but these were not good enough to obtain the exact measure function.

For $k=1$, there is no problem about the Hausdorff measure of the path

(4) *Added in proof.* This conjecture (6.3) has since been proved by D. Ray.

set since this is just an interval on the line. The methods used here can be applied to find the distribution of $T_1(a, X, \omega)$, though again the computational difficulties are formidable.

References

1. R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, Vols. 1, 2, Springer, Berlin, 1937.
2. D. A. Darling and M. Kac, *On occupation times for Markov processes*, Trans. Amer. Math. Soc. **84** (1957), 444–458.
3. J. L. Doob, *Stochastic processes*, Wiley, New York, 1953.
4. A. Dvoretzky and P. Erdös, *Some problems on random walk in space*, Proceedings of the Second Berkeley Symposium of Mathematical Statistics and Probability, 1951, 353–367.
5. P. Erdös, *On the law of the iterated logarithm*, Ann. of Math. **43**, 1942, 419–436.
6. P. Erdös and S. J. Taylor, *Some problems concerning the structure of random walk paths*, Acta Math. Acad. Sci. Hungar. **11** (1960), 137–162.
7. ———, *On the Hausdorff measure of Brownian paths in the plane*, Proc. Cambridge Philos. Soc. **57** (1961), 209–222.
8. F. Hausdorff, *Dimension und äusseres Mass*, Math. Ann. **79** (1919), 157–179.
9. I. I. Hirschman and D. V. Widder, *The convolution transform*, Princeton Univ. Press, Princeton, N. J., 1955.
10. M. Kac, *On some connections between probability theory and differential and integral equations*, Proceedings of the Second Berkeley Symposium of Mathematical Statistics and Probability, 1951, 180–215.
11. J. Kiefer, *K-sample analogues of the Kolmogorov-Smirnov and Cramer-v. Mises tests*, Ann. Math. Statist. **30** (1959), 420–447.
12. P. Lévy, *La mesure de Hausdorff de la courbe du mouvement brownien*, Giorn. Ist. Ital. Attuari **16** (1953), 1–37.
13. C. A. Rogers and S. J. Taylor, *The analysis of additive set functions in Euclidean space*, Acta Math. **101** (1959), 273–302.
14. ———, *Functions continuous and singular with respect to a Hausdorff measure*, Mathematika **8** (1961), 1–31.
15. M. Rosenblatt, *On a class of Markov processes*, Trans. Amer. Math. Soc. **71** (1951), 120–135.
16. S. J. Taylor, *The Hausdorff α-dimensional measure of Brownian paths in n-space*, Proc. Cambridge Philos. Soc. **49** (1953), 31–39.
17. ———, *On the connection between generalized capacities and Hausdorff measures*, Proc. Cambridge Philos. Soc. **57** (1961), 524–531.
18. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, Cambridge, 1944.

Cornell University,
Ithaca, New York